e content for students of patliputra university

B. Sc. (Honrs) Part 2paper 3

Subject:Mathematics

Title/Heading of topic: Infinite series

By Dr. Hari kant singh

Associate professor in mathematics

Rrs college mokama patna

INFINITE SERIES

2.1 Sequences: A sequence of real numbers is defined as a function $f: \boldsymbol{N} \rightarrow \boldsymbol{R}$,where $\boldsymbol{N}$ is a set of natural numbers and **R** is a set of real numbers. A sequence can be expressed as$\langle f_1, f_2, f_3, \ldots \ldots, f_n, \ldots . \rangle$or $\langle f_n \rangle$. For example $\langle \frac{1}{n} \rangle = \langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots \ldots . \rangle$ is a sequence.

Convergent sequence: A sequence $\langle u_n \rangle$ converges to a number l, if for given $\varepsilon > 0$, there exists a positive integer m depending on ε , such that $|u_n - l| < \varepsilon \; \forall \, n \geq m$.

Then l is called the limit of the given sequence and we can write

$\lim_{n \to \infty} u_n = l$ or $u_n \rightarrow l$

2.2 Definition of an Infinite Series

An expression of the form $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is known as the infinite series of real numbers, where each $u_n$is a real number. It is denoted by$\sum_{n=1}^{\infty} u_n$.

For example $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$is an infinite series.

Convergence of an infinite series

Consider an infinite series $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots$

Let us define$S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3$, ……..,

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$ and so on .

Then the sequence $\langle S_n \rangle$ so formed is known as the sequence of partial sums (S.O.P.S.) of the given series.

Convergent series: A series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$ converges if the sequence $\langle S_n \rangle$ of its partial sums converges i.e. if $\lim_{n\to\infty} S_n$ exists. Also if $\lim_{n\to\infty} S_n = S$ then S is called as the sum of the given series .

Divergent series: A series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$ diverges if the sequence $\langle S_n \rangle$ of its partial sums diverges i.e. if $\lim_{n\to\infty} S_n = +\infty \ or - \infty$.

Example 1 Show that the Geometric series $\sum_{n=1}^{\infty} r^{n-1} = 1 + r + r^2 + r^3 + \ldots..$, where $r > 0$, is convergent if $r < 1$ and diverges if $r \geq 1$.

Solution: Let us define $S_1 = 1$, $S_2 = 1 + r$, $S_3 = 1 + r + r^2$, ...,

$$S_n = 1 + r + r^2 + \cdots + r^{n-1}$$

Case 1: $r < 1$

$$\text{Consider } \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r} - \lim_{n\to\infty} \frac{r^n}{1-r}$$

$$= \frac{1}{1-r} \quad \left(\text{As } \lim_{n\to\infty} r^n = 0 \text{ if } |r| < 1\right)$$

Since $\lim_{n\to\infty} S_n$ is finite $\therefore$ the sequence of partial sums i.e. $\langle S_n \rangle$ converges and hence the given series converges.

Case2: $r > 1$

$$\text{Consider } \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r^n-1}{r-1} = \lim_{n\to\infty} \frac{r^n}{1-r} - \frac{1}{r-1}$$

$\to \infty$ (As $r^n \to \infty$ if $r > 1$)

Since $\langle S_n \rangle$ diverges and hence the given series diverges.

Case2: $r = 1$

$$\text{Consider } S_n = 1 + r + r^2 + \cdots + r^{n-1}$$

$$= 1 + 1 + 1 + 1 + \cdots .. + 1 = n \Rightarrow \lim_{n\to\infty} S_n = \infty$$

Since $\langle S_n \rangle$ diverges and hence the given series diverges.

Positive term series

An infinite series whose all terms are positive is called a positive term series.

p-series: An infinite series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots (p > 0)$is called p-series.

It converges if$p > 1$ and diverges if $p \leq 1$.

For example:

1.$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$ converges (As $p = 3 > 1$)

2.$\sum_{n=1}^{\infty} \frac{1}{n^{5/2}} = \frac{1}{1^{5/2}} + \frac{1}{2^{5/2}} + \frac{1}{3^{5/2}} + \cdots$ converges $\left(\text{As } p = \frac{5}{2} > 1\right)$

3.$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \frac{1}{1^{1/2}} + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \cdots$ converges $\left(\text{As } p = \frac{1}{2} < 1\right)$

Necessary condition for convergence:

If an infinite series $\sum_{n=1}^{\infty} u_n$ is convergent then $\lim_{n \to \infty} u_n = 0$. However , converse need not be true.

Proof: Consider the sequence $\langle S_n \rangle$of partial sums of the series$\sum_{n=1}^{\infty} u_n$.

We know that $S_n = u_1 + u_2 + u_3 + \cdots\cdots\cdots\cdots + u_n$

$$= u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n$$

$$\Rightarrow S_{n-1} = u_1 + u_2 + u_3 + \cdots\cdots\cdots + u_{n-1}$$

Now $S_n - S_{n-1} = u_n$

Taking limit $n \to \infty$, we get

$$\lim_{n \to \infty} (S_n - S_{n-1}) = \lim_{n \to \infty} u_n$$

$\Rightarrow \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} u_n$ ....(1)

As $\sum_{n=1}^{\infty} u_n$ is convergent $\therefore$ sequence $\langle S_n \rangle$ of its partial sums is also convergent.

Let $\lim_{n\to\infty} S_n = l$, then $\lim_{n\to\infty} S_{n-1} = l$

Substituting these values in equation (1), we get $\lim_{n\to\infty} u_n = 0$.

To show that converse may not hold , let us consider the series$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$.

Here $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

But$\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series (As $p = 1$)

Corollary: If $\lim_{n\to\infty} u_n \neq 0$, then $\sum_{n=1}^{\infty} u_n$cannot converge.

Example 2 Test the convergence of the series $\sum_{n=1}^{\infty} cos\frac{1}{n}$

Solution: Here $u_n = cos\frac{1}{n} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} cos\frac{1}{n} = 1 \neq 0$

Hence the given series is not convergent.

Example 3 Test the convergence of the series $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$

Solution: Here $u_n = \sqrt{\frac{n}{n+1}} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1 \neq 0$$

Hence the given series is not convergent.